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Small Random Perturbations of Finite- and Infinite-Dimensional Dynamical Systems: Unpredictability of Exit Times

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We apply previous results on the pathwise exponential loss of memory of the initial condition for stochastic differential equations with small diffusion to the problem of the asymptotic distribution of the first exit times from an attracted domain. We show under general hypotheses that the suitably rescaled exit time converges in the zero-noise limit to an exponential random variable. Then we extend the results to an infinite-dimensional case obtained by adding a small random perturbation to a nonlinear heat equation.

KEY WORDS: Small random perturbations; dynamical systems; exit times.

1. INTRODUCTION

In this paper we consider some problems arising in the study of small random perturbations of dynamical systems.

In particular we deal with a class of Ito equations in $\mathbf{R}^n$ of the form

$$\begin{aligned} dx_t^\varepsilon &= b(x_t^\varepsilon)\, dt + \varepsilon\, dw_t \\ x_0^\varepsilon &= x_0 \end{aligned} \tag{1.1}$$

Equations like (1.1), obtained by adding a Brownian noise term to a deterministic time evolution, appear in several branches of natural sciences and many interesting problems arise in their study.

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In particular, we will be interested in some aspects of the behavior of x_t^ε in the limit $\varepsilon \to 0$: we will study the asymptotic behavior, for $\varepsilon \to 0$, of the exit time from a bounded domain $G \subset \mathbf{R}^n$:

$$\tau_G^\varepsilon = \inf\{t > 0: x_t^\varepsilon \notin G\}$$

Of course one expects that for small ε, x_t^ε will approach, in some sense, the solution x_t^0 of the deterministic equation

$$\begin{aligned} \frac{dx_t^0}{dt} &= b(x_t^0) \\ x^0(0) &= x_0 \end{aligned} \tag{1.2}$$

The case in which G is invariant in the future with respect to the time evolution given by Eq. (1.2) appears to be particularly significant. The exit from G of the diffusion process x_t^ε, in this case, is a very rare event in which the stochastic motion goes against the drift b and from a probabilistic point of view it represents a typical large-deviation phenomenon.

This kind of problem has been extensively studied by Freidlin and Ventzel in a fundamental series of papers.[1-3] In particular, they studied $\mathbf{E}(\tau_G^\varepsilon)$, showing that it diverges exponentially in $1/\varepsilon^2$. A more specific question that goes beyond the analysis of Ventzell and Freidlin concerns the asymptotic form of the distribution of τ_G^ε as $\varepsilon \to 0$.

In many interesting cases one can prove that $\tau_G^\varepsilon/\mathbf{E}(\tau_G^\varepsilon)$ converges in distribution to an exponential random variable of mean one. This feature can be intuitively interpreted as a consequence of the fact that, for the exit from G, one needs many practically independent, very unlikely, similar attempts. The asymptotic exponentially of the exit time shows the "unpredictability" of the event. This is in general a nontrivial and deep probabilistic result; in this paper it is proved under general conditions for finite- [see (1.1)] and infinite-dimensional models [see (1.3)].

This result is also relevant from a physical point of view in the context of the so-called "pathwise approach to the metastability."

This kind of approach has been introduced in ref. 4 to describe, from a dynamical point of view, the phenomenon of metastability that arises in connection with some first-order phase transitions. In this framework, as an alternative to the so-called "evolution of the ensembles," the behavior of each typical path is studied and in this way one can give sense to what is called metastable behavior for a general stochastic dynamics.

Let us recall some previous results. Day[5] shows the asymptotic exponentially of $\tau_G^\varepsilon/\mathbf{E}(\tau_G^\varepsilon)$ when G is a bounded domain completely attracted by a unique, asymptotically stable equilibrium point with respect to the deterministic evolution given by Eq. (2.2). The author considers the more general case of nonconstant diffusion.

In ref. 6 the authors consider the case when $b = -\nabla U$, where U is a two-well potential with only three critical points: the absolute minimum q, the local minimum p, and a saddle point r; they analyze the so-called "tunnelling phenomenon," namely the transition from the vicinities of p to the vicinities of q induced by the noise.

In particular, they show the asymptotic exponentiality of the suitably rescaled tunneling time. The result of ref. 6 represents an extension of the one of Day, since one has to deal with a domain G containing, besides a stable equilibrium point, also a saddle point.

All these results use in a substantial way some analytical results based on a rather detailed analysis of the infinitesimal generator of the diffusion process x_t^ε. This feature makes, for example, rather difficult the extension of the results to infinite-dimensional cases.

In the present paper we present a very general approach to the analysis of the asymptotics of the first exit time from a domain G. Our strategy is purely probabilistic and it applies equally well to the finite- and infinite-dimensional cases. The approach is based on the new results about the exponential joining in time of stochastic trajectories starting from different points but subjected to the same noise proved in ref. 7.

The above result is very simple if, e.g., $b(x) = -\nabla U(x)$ and U is a strictly convex function bounded from below, but it is by no means trivial when b has several equilibria. The proof of the exponential loss of memory of the initial conditions in ref. 7 required in the general case, besides the Ventzel and Freidlin techniques, a detailed analysis of the "typical paths" of the stochastic process.

The infinite-dimensional case that will be studied later in this work is the model introduced in ref. 8. It is described by the formal stochastic partial differential equation given by

$$\begin{aligned} \partial_t u &= \partial_{xx} u - V'(u) + \varepsilon\alpha \\ u(x, 0) &= u_0(x) \\ u(0, t) &= u(L, t) = 0 \end{aligned} \tag{1.3}$$

where $x \in [0, L]$, $t \geqslant 0$, $V(u) = (\lambda/4)\, u^4 - (\mu/2)\, u^2$, and $\alpha(x, t)$ is the standard space-time white noise, i.e., the Gaussian random field with zero mean and covariance given by

$$\mathbf{E}(\alpha(x, t)\, \alpha(x', t')) = \delta(x - x')\, \delta(t - t')$$

Equation (1.3) can be viewed as a random perturbation of an infinite-dimensional dynamical system of gradient type: in fact, we can write, for $\varepsilon = 0$,

$$\partial_t \tilde{u} = \partial_{xx} \tilde{u} - V'(\tilde{u}) = -\delta S(\tilde{u})/\delta \tilde{u} \tag{1.4}$$

where

$$S(u) = \int_0^L dx\, [\tfrac{1}{2}(\partial_x u)^2 + V(u)]$$

Another possible physical meaning of Eq. (1.3) concerns the time evolution of the magnetization profile for a mean field one-dimensional ferromagnetic model (for more details see ref. 9).

One can see that for suitable values of $(\mu L)^{1/2}$ the deterministic gradient flow given by Eq. (1.5) admits several equilibria, two of which are stable.

In refs. 8 and 9 the authors study the tunneling phenomenon, namely the transition induced by the noise between the two stable equilibria.

In Section 3 we prove the exponential joining of the random fields starting from different initial configurations along the same lines of ref. 7.

In Section 4, using this result, we provide a simple proof of the rescaled tunneling time.

2. THE FINITE-DIMENSIONAL CASE

Let $X_t^\varepsilon(x)$ be the Markov process solution of the Ito stochastic differential equation in $\mathbf{R}^n$:

$$\begin{aligned} dX_t^\varepsilon &= b(X_t^\varepsilon)\, dt + \varepsilon\, dw_t \\ X_0^\varepsilon &= x \end{aligned} \tag{2.1}$$

where $\{w_t\}$ is the ordinary Brownian motion in $\mathbf{R}^n$ starting at the origin, ε is a positive constant, and the drift term $b(x)$ is assumed to be a smooth vector field satisfying the hypotheses listed below.

For notational convenience we will denote by $X_{t,t_0}^\varepsilon(x)$ the solution at time t of the equation

$$X_{t,t_0}^\varepsilon(x) = x + \int_{t_0}^t ds\, b(X_{s,t_0}^\varepsilon(x)) + \varepsilon(w_t - w_{t_0})$$

As usual $X_{t,0}^\varepsilon(x)$ will be denoted by $X_t^\varepsilon(x)$.

The first assumption on $b(x)$ makes sure that the process X_t^ε solution of (2.1) admits a unique, smooth invariant measure μ^ε (see, e.g., ref. 10):

hp1: There exist $R_0 > 0$ and $a > 0$ such that if $n(x, R)$ denotes the outward normal to the surface of the sphere $B_R = B_R(0)$, centered at 0 of radius R, at the point x, then

$$\sup_{R > R_0} \sup_{x \in \partial B_R} \mathbf{b}(x) \cdot \mathbf{n}(x, R) < -a < 0$$

Thus, the drift b is confining and the process X_t^ε will spend most of its time in the ball B_{R_0}. We also assume that $\sup_{x \in \mathbf{R}^n} |\nabla b(x)| < K$ for some $K > 0$. The next assumption concerns the long-time behavior of the dynamical system

$$dX_t/dt = b(X_t) \tag{2.2}$$

Following Ventzel and Freidlin,[1-3] let, for any continuous function ϕ: $[0, T] \to \mathbf{R}^n$,

$$I_{0,T}(\phi) = \int_0^T dt\, \|\dot{\phi}_t - b(\phi_t)\|^2 \tag{2.3}$$

if the integral exists and $I_{0,T}(\phi) = \infty$ if not. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$. For $x, y \in \mathbf{R}^n$ we also set

$$V(x, y) = \inf_{\phi:\, \phi(0) = x,\, \phi(T) = y} I_{0,T}(\phi) \tag{2.4}$$

We establish an equivalence relation in $\mathbf{R}^n$ in the following way:

$$x \approx y \leftrightarrow x = y \qquad \text{or} \qquad V(x, y) = V(y, x) = 0 \tag{2.5}$$

An equivalence class K is said to be "stable" if

$$V(x, y) > 0 \qquad \forall x \in K, \quad \forall y \notin K \tag{2.6}$$

and "unstable" if it is not stable.

hp2: There exist finitely many compacta $K_1, \ldots, K_N$ such that:

(i) For any two points x, y in K_i, $x \approx y$.

(ii) If $x \in K_i$ and $y \notin K_i$, then $x \not\approx y$.

(iii) Every ω-limit set of the system (2.2) is contained in some K_i.

(iv) Exactly the first $l \leqslant N$ compacta are stable and they consist of one point (fixed point).

It is easy to see that $V(x, y)$ attains the same value V_{ij} for all $x \in K_i$, $y \in K_j$. In terms of the numbers V_{ij} the stability of a compactum K_i can be also stated as follows[1]:

$$K_i \quad \text{is stable iff} \quad V_{ij} > 0 \qquad \forall j \neq i \tag{2.7}$$

Let now F_t be a family of smooth map, from $\mathbf{R}^n$ to $\mathbf{R}^n$ and let

$$L(F_t, x) = \lim_{\varepsilon \to 0} \sup_{\|x-y\| < \varepsilon} \frac{\|F_t(x) - F_t(y)\|}{\|x - y\|} \tag{2.8}$$

hp3: There exist a number m_0 and, for each stable compactum K_i, $i = 1,..., l$, a neighborhood $\bar{A}_i$ such that

$$\sup_{x \in A_i} L(X_t, x) < e^{-m_0 t} \qquad \forall t > 0$$

where $X_t(x)$ is the solution of (2.2) starting at x.

This last assumption is clearly satisfied if the Jacobian matrices $\partial b_i/\partial x_j$ at the stable fixed points have eigenvalues with negative real part smaller than $-m_0$. Under hp1–3 the following basic result was proved in ref. 7.

Let x_i be a stable fixed point and let

$$d_i = \sup\{\delta > 0\colon \sup_{x \in B_\delta(x_i)} L(X_t, x) \leqslant e^{-m_0 t}\ \forall t > 0\}$$

Using hp3, one finds that the quantity $d = \min_{1 \leqslant i \leqslant l} d_i$ is strictly greater than zero and we will denote by A_i the set $B_d(x_i)$; we define

$$A = \bigcup_{i=1}^{l} A_i, \qquad C_i = B_{d/2}(x_i), \qquad C = \bigcup_{i=1}^{l} C_i$$

$$C_{0i} = B_{d/4}(x_i), \qquad C_0 = \bigcup_{i=1}^{l} C_{0i}$$

with the above notation our results read as follows.

Theorem 2.1. For any $m < m_0$ there exist positive constants K, K', t_0, ε_0 such that for any $\varepsilon < \varepsilon_0$ and any $t > t_0$:

(a) $\quad P(X_t^\varepsilon(x) \in C\ \forall x \in C \text{ and } \sup_{x \in C} L(X_t^\varepsilon, x) \leqslant e^{-mt}) \geqslant 1 - e^{-K/\varepsilon^2}$

(b) $\quad P(\sup_{x \in C} L(X_t^\varepsilon, x) \leqslant e^{-mt}\ \forall t > t_0) \geqslant 1 - e^{-K'/\varepsilon^2}$

where $X_t^\varepsilon(x)$ is the solution of (2.1) starting at x.

Remark 1. It is easy to verify that if

$$\sup_{x \in C} L(X_t^\varepsilon, x) \leqslant e^{-mt}$$

then

$$\sup_i \sup_{x, y \in C_i} \frac{\|X_t^\varepsilon(x) - X_t^\varepsilon(y)\|}{\|x - y\|} \leqslant e^{-mt}$$

Remark 2. Actually, Theorem 2.1 in ref. 7 was also proved under more general hypotheses on the drift $b(x)$. For example, the gradient case $b(x) = -\nabla U(x)$, with $U(x)$ having quartic minima (so that hp3 does not hold), was also analyzed. In this case the exponent m become infinitesimal as $\varepsilon \to 0$. We refer the reader to ref. 7 for a more complete and critical discussion of our hypotheses.

We now turn to the main problem to be discussed in this section. Let $G \subset \mathbf{R}^n$ be a compact set with a smooth boundary such that:

hp4:

$$G \cap C = C_1 \cup C_2 \cup \cdots \cup C_{l'}, \quad l' \leqslant l, \qquad \text{and} \qquad \partial G \cap \bar{C} = \varnothing$$

hp5: Let $V = \max_{n,m \leqslant l'} V_{n,m}$ and let $V_G = \min_{1 \leqslant n \leqslant l'} \min_{y \in \partial G} V(x_n, y)$. Then $V_G > V$.

Let finally $\tau_G(x) = \inf\{t \geqslant 0; X_t^\varepsilon(x) \notin G\}$. Under hp1–5 our result on the asymptotic distribution of $\tau_G(x)$ as $\varepsilon \to 0$ reads as follows.

Theorem 2.2. Let $\beta \equiv \beta(\varepsilon)$ be such that

$$\sup_{x \in G \cap C} \mathbf{P}(\tau_G(x) > \beta(\varepsilon)) = e^{-1}$$

Then

(i) $$\lim_{\varepsilon \to 0} \frac{\mathbf{E}[\tau_G(x)]}{\beta} = 1 \qquad \forall x \in G \cap C$$

(ii) $$\lim_{\varepsilon \to 0} \mathbf{P}(\tau_G(x) > \beta t) = e^{-t} \qquad \forall t \geqslant 0, \quad \forall x \in G \cap C$$

Remark 3. It is an easy consequence of the Ventzel and Freidlin theory that there exists a constant $h > 0$ such that $\beta > \exp(h/\varepsilon^2)$ for ε small enough.

Remark 4. If we have $\mathbf{b}(x) \cdot \mathbf{n}(x) < 0$ for any $x \in \partial G$, where $\mathbf{n}(x)$ denotes the outward normal to the boundary of G at x, then the above results extend to all x in the interior of G.

The strategy of the proof is basically that of ref. 6, but the result of Theorem 2.1 simplify considerably the whole argument. The main technical estimate, which is the hard part of the proof, is contained in the following lemma, whose proof is postponed to that of Theorem 2.2. For notational convenience we will denote by $o(1)$ any function of ε going to zero as $\varepsilon \to 0$.

Lemma 2.1. For any $t > 0$

$$\sup_{x, y \in G \cap C} |P(\tau_G(x) > \beta t) - P(\tau_G(y) > \beta t)| = o(1)$$

Proof of Theorem 2.2. We start with (ii). Using the strong Markov property, we will show that $f_\varepsilon(x, t) \equiv P(\tau_G(x) > \beta t)$ satisfies

$$f_\varepsilon(x, t+s) = f_\varepsilon(x, t) f_\varepsilon(x, s) + o(1) \qquad \forall x \in G \cap C \tag{2.9}$$

The above estimate suffices to prove (ii). From (2.9) it follows in fact that for any $x \in G \cap C$: (a) The family $\{f_\varepsilon(x, t)\}$ is tight as $\varepsilon \to 0$. (b) Let $f^*(x, t)$ be any limit point (in distribution) of the family $\{f_\varepsilon(x, t)\}$. Then $f^*(x, t) = e^{-t}$.

(a) Tightness follows immediately: in fact, by applying (2.9) inductively k times, we get

$$f_\varepsilon(x, 2^k) - [f_\varepsilon(x, 1)]^k = o(1) \tag{2.10}$$

and

$$[f_\varepsilon(x, 2^{-k})]^k - f_\varepsilon(x, 1) = o(1) \tag{2.11}$$

This means that for any $\delta > 0$ we can find ε_0, k_0 such that

$$f_\varepsilon(x, 2^{-k_0}) > 1 - \delta; \qquad f_\varepsilon(x, 2^{k_0}) < \delta \tag{2.12}$$

whenever $\varepsilon < \varepsilon_0$. Clearly, (2.12) implies tightness.

(b) Convergence to the exponential law is obvious from (2.9) if we use Lemma 2.1 to derive the normalization condition

$$f_\varepsilon(x, 1) = e^{-1} - [\sup_{x' \in G \cap C} P(\tau_G(x') > \beta) - f_\varepsilon(x, 1)] = e^{-1} + o(1) \tag{2.13}$$

Thus we have to prove (2.9). By the Markov property we write

$$P(\tau_G(x) > \beta(t+s)) = E\chi(\tau_G(x) > \beta s)\, E\chi(\tau_G(X^\varepsilon_{\beta s}(x)) > \beta t) \tag{2.14}$$

By Theorem 2.1 we have that since $X^\varepsilon_{\beta s}(x) \in G$, then, with large probability, $X^\varepsilon_{\beta s}(x) \in C \cap G$, i.e.,

$$f_\varepsilon(x, t+s) = E\chi(\tau_G(x) > \beta s)\, \chi(X^\varepsilon_{\beta s}(x) \in C \cap G)\, E\chi(\tau_G(X^\varepsilon_{\beta s}(x)) > \beta t) + o(1) \tag{2.15}$$

Using now Lemma 2.1 and again Theorem 2.1, we get

$$f_\varepsilon(x, t+s) = f_\varepsilon(x, t)\, f_\varepsilon(x, s) + o(1)$$

This concludes the proof of part (ii).

To prove that $\lim_{\varepsilon \to 0} [E\{\tau_G(x)\}/\beta] = 1$, $\forall x \in G \cap C$, we write

$$\frac{E\{\tau_G(x)\}}{\beta} = \int_0^\infty dt\, P(\tau_G(x) > \beta t) \tag{2.16}$$

In order to perform the limit $\varepsilon \to 0$ inside the integral we need a uniform control as $\varepsilon \to 0$ on the tail of the distribution of $\tau_G(x)/\beta$.

Let $g(t) = \sup_{x \in G} P(\tau_G(x) > \beta t)$. Then, by the strong Markov property [see (2.14)], we get

$$g(t+s) \leqslant g(t)\, g(s) \tag{2.17}$$

that is, $g(2^k) \leqslant g(2)^{k-1}$. We estimate $g(2)$. Given $x \in G$, let

$$\sigma(x) = \inf(t \geqslant 0: X_t^\varepsilon(x) \in C) \tag{2.18}$$

Then we can write, again by the strong Markov property,

$$\begin{aligned} g(2) &\leqslant \sup_{x \in G} P(\sigma(x) > \beta) + \sup_{x \in G} E\chi(\sigma(x) < \beta)\, E_{X^\varepsilon_{\sigma(x)}} \chi(\tau_G > 2\beta - \sigma(x)) \\ &\leqslant \sup_{x \in G} P(\sigma(x) > \beta) + \sup_{x \in G \cap C} P(\tau_G > \beta) \end{aligned} \tag{2.19}$$

The second term is equal to e^{-1} by the definition of β, while the first one is $o(1)$ by standard results of Ventzel and Freidlin. In conclusion,

$$g(2) \leqslant e^{-1} + o(1) \tag{2.20}$$

By induction from Eq. (2.20) we get that g is $L^1(\mathbf{R}^+)$ and so, by the dominated convergence theorem, we deduce that we can perform the limit $\varepsilon \to 0$ inside the integral in (2.16) and the result follows from (ii).

Remark 5. From the above argument and without using the convergence of $P(\tau_G > \beta t)$ to the exponential law it follows that

$$\beta > \text{const} \cdot E(\tau_G)$$

for a suitable constant independent of ε. Using the results of Ventzel and Freidlin, this implies that

$$\liminf_{\varepsilon \to 0} 2\varepsilon^2 \ln(\beta) \geqslant V_G$$

Proof of Lemma 2.1. Let

$$\tau_j = \inf(t \geqslant 0;\ X_t^\varepsilon(x) \in C_j), \qquad j = 1,..., l' \tag{2.21}$$

and let $T_0 = e^{(V + V_G)/4\varepsilon^2}$.

The theory of Ventzel and Freidlin tells us that

$$\limsup_{\varepsilon \to 0} 2\varepsilon^2 \ln E(\tau_j(x)) \leqslant V \qquad \forall x \in C \cap G \tag{2.22}$$

Thus, the time T_0 is very large compared with the typical time scale of τ_j, $j = 1,..., l'$, but, using Remark 4, it is very small compared with the time scale of τ_G. This observation suggests that we write for a given j and for any x in $G \cap C$

$$\begin{aligned} P(\tau_G(x) > \beta t) &\leqslant P(\tau_G(x) > \beta t;\ \tau_j(x) < T_0 t) + P(\tau_j(x) > T_0 t) \\ &\leqslant P(\tau_G(x) > \beta t;\ \tau_j(x) < T_0 t) + E(\tau_j(x))/T_0 t \end{aligned} \tag{2.23}$$

where we used the Chebyshev inequality. Thus, by (2.22) and the definition of T_0 the term in the lhs of (2.23) and the first term in the rhs differ by $o(1)$. Now, for any x in $G \cap C$, we compare $P(\tau_G(x) > \beta t)$ with $P(\tau_G(x_j) > \beta t)$. By the strong Markov property and (2.23) we have

$$\begin{aligned} o(1) + \inf_{x \in C_j} P(\tau_G(x) > \beta t) &\leqslant P(\tau_G(x) > \beta t) \\ &\leqslant \sup_{x \in C_j} P(\tau_G(x) > \beta(1 - T_0/\beta) t) + o(1) \end{aligned}$$

which gives

$$\begin{aligned} &|P(\tau_G(x) > \beta t) - P(\tau_G(x_j) > \beta t)| \\ &\quad \leqslant \sup_{x \in C_j} |P(\tau_G(x) > \beta t) - P(\tau_G(x_j) > \beta t)| \\ &\qquad + \sup_{x \in C_j} |P(\tau_G(x) > \beta t) - P(\tau_G(x) > \beta(1 - T_0/\beta t) t)| + o(1) \end{aligned} \tag{2.24}$$

The second term in the rhs of (2.23) is estimated by

$$\begin{aligned} &\sup_{x \in C_j} P(\beta(1 - T_0/\beta) t \leqslant \tau_G(x) \leqslant \beta t) \\ &\quad \leqslant \sup_{x \in C_j} P(x^\varepsilon_{\beta(1 - T_0/\beta)t}(x) \notin C \cap G) + \sup_{x \in C \cap G} P(\tau_G(x) < T_0 t) = o(1) \end{aligned} \tag{2.25}$$

because of Theorem 2.1 of ref. 7 and of the Ventzel and Freidlin result

$$\sup_{x \in C \cap G} P(\tau_G(x) < T_0 t) = o(1)$$

So, since $x \in G \cap C$ was completely arbitrary, we have reduced the proof of Lemma 2.1 to the problem of showing that the first term in the rhs of (2.24) is $o(1)$. It is at this point that the result of Theorem 2.1 plays a crucial role. In fact, by using Theorem 2.1 together with Remark 1 we have that with large probability the two paths $X_t^\varepsilon(x)$, $X_t^\varepsilon(x_j)$, $x \in C_j$, will join exponentially fast and therefore the two exit times $\tau_G(x)$ and $\tau_G(x_j)$ will be almost the same.

More precisely, we have

$$\begin{aligned} \sup_{x \in C_j} & |P(\tau_G(x) > \beta t) - P(\tau_G(x_j))| \\ & \leqslant \sup_{x \in C_j} E(\chi(\tau_G(x) > \beta t)\, \chi(1/\varepsilon^2 < \tau_G(x_j) < \beta t) \\ & \quad \times \chi(|x^\varepsilon_{\tau_G(x_j)}(x) - x^\varepsilon_{\tau_G(x_j)}(x_j)| < |x - x_j|\, e^{-m\tau_G(x_j)}) \\ & \quad + (\text{same term with } x \text{ and } x_j \text{ interchanged}) + o(1) \end{aligned} \tag{2.26}$$

where $o(1)$ represents the probability of those paths such that either the conditions of Theorem 2.1 are violated or $\min(\tau_G(x_j), \tau_G(x)) < 1/\varepsilon^2$.

The first two terms in the rhs of (2.26) are estimated in the same way and we therefore treat only the first one. By making again an error $o(1)$ [see (2.25)], we can substitute the characteristic function $\chi(1/\varepsilon^2 < \tau_G(x_j) < \beta t)$ with $\chi(1/\varepsilon^2 < \tau_G(x_j) < \beta t - \varepsilon t)$. Thus, by the strong Markov property our estimate reduces to the estimate of

$$\sup_{x \in G;\ \mathrm{dist}(x, \partial G) < \exp(-m/\varepsilon^2)} P(\tau_G(x) > \varepsilon t) \tag{2.27}$$

To show that this last quantity is $o(1)$, we proceed as follows. Let $\partial^\varepsilon = (x \notin G;\ \mathrm{dist}(x, G) = \exp(-m/\varepsilon^2))$; it is clear that for ε sufficiently small, ∂^ε is a smooth surface (e.g., C^2) enclosing G. Let also

$$\sigma(x) = \inf\{t \geqslant 0;\ \varepsilon w_t(x) \in \partial^\varepsilon;\ x \in G\} \tag{2.28}$$

where $w_t(x)$ is the Brownian motion starting at x. Rather standard estimates on the Brownian motion show that

$$\sup_{x \in G;\ \mathrm{dist}(x, \partial G) < \exp(-m/\varepsilon^2)} P\left(\sigma(x) > \exp\left[-\frac{3m/2}{\varepsilon^2}\right]\right) = o(1) \tag{2.29}$$

This is because on a time scale $\exp[-(3m/2)/\varepsilon^2]$ the Brownian motion εw_t has fluctuations of order $\varepsilon \exp[-(3m/4)/\varepsilon^2] \gg 2\exp(-m/\varepsilon^2)$ for ε small enough. Furthermore,

$$|x^\varepsilon_{\sigma(x)}(x) - \varepsilon w_{\sigma(x)}(x)| \leqslant k\sigma(x) \tag{2.30}$$

for some $k>0$ so that

$$\tau_G(x) < \sigma(x)$$

if $\sigma(x) < \exp[-(-3m/2)/\varepsilon^2]$ and ε is small enough. This, together with (2.29), proves that (2.27) is $o(1)$.

3. THE INFINITE-DIMENSIONAL CASE

3.1. The Model and Main Results

In this section we start the analysis of the problem discussed in the previous section for a stochastic partial differential equation. We want to follow the strategy that has been used in the finite-dimensional case and therefore we first have to prove the analogue of Theorem 2.1 in the new context.

The model that we shall consider here is the one introduced by Faris and Jona-Lasinio.[8] It concerns a nonlinear heat equation with noise in one dimension that can formally be written as

$$\begin{aligned} &\partial_t u^\varepsilon = \partial_{xx} u^\varepsilon - V'(u^\varepsilon) + \varepsilon\alpha(t, x) \\ &u^\varepsilon(x, 0) = u_0(x); \qquad u^\varepsilon(0, t) = u^\varepsilon(L, t) = 0 \quad \forall t > 0 \end{aligned} \tag{3.1}$$

where $x \in [0, L]$, $t \geqslant 0$, $V(u) = (\lambda/4)\, u^4 - (\mu/2)\, u^2$, λ, μ, and ε are positive, and $\alpha(x, t)$ is the standard space-time white noise, i.e., the Gaussian random field with zero mean and covariance given by

$$E(\alpha(x, t)\, \alpha(x', t')) = \delta(x - x')\, \delta(t - t') \tag{3.2}$$

The standard way to give a precise meaning to Eq. (3.1) is to transform it into the integral equation

$$u^\varepsilon = -GV'(u^\varepsilon) + \varepsilon W_t + g u_0 \tag{3.3}$$

where we have the following.

1. $g(x, y, t) = e^{t\Delta_D}(x, y) = \frac{2}{L} \sum_{n \geqslant 1} e^{-n^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right)$

with Δ_D the Dirichlet Laplacian on $L^2[0, L]$.

2. G is the operator that solves the inhomogeneous heat equation with zero initial condition. G has the expression

$$G(x, t, y, s) = g(x, y, t - s)\, \theta(t - s)$$

where θ is the Heaviside function.

3. $W(x, t) \equiv W_t(x)$, formally given by $W_t(x) = (G\alpha)(x, t)$ is the Gaussian field with covariance

$$E(W_t(x)\, W_s(y)) = \int_0^{t \wedge s} d\tau \int_0^L dz\, G(x, t, z, \tau)\, G(y, s, z, \tau)$$

For more details see ref. 8.

Notation. We will denote by $C_{D,u_0}([0, L] \times [0, T])$ the space of real, continuous functions on $[0, L] \times [0, T]$ that satisfy Dirichlet boundary conditions on $[0, L]$ and such that $u(x, 0) = u_0(x)$. By C_D we will denote the space $C_D([0, L])$. The uniform norm will be denoted by $\|\cdot\|_\infty$.

Next we recall some results concerning the solution of (3.1).

Proposition 3.1. The Gaussian random field $W_t(x)$ is such that, $\forall t > 0$, $x \to W_t(x)$ is a *Hölder* continuous function with exponent $1/2 - \delta < \alpha' < 1/2$, $\forall \delta > 0$, satisfying the Dirichlet boundary conditions.

Proposition 3.2. $\forall T > 0$, $\forall u_0 \in CD_D$, $\forall W \in C_{D,0}([0, L] \times [0, T])$ there exists a unique solution u of (3.3) in $C_{D,u_0}([0, L] \times [0, T])$. Furthermore, the solution u depends continuously in the uniform norm on $z = W + u$.

In order to describe the qualitative behavior in time of the random field $u(x, t)$ it is important to have a detailed description of the behavior of the associated unperturbed infinite-dimensional dynamical system:

$$\partial_t u^0 = \partial_{xx} u^0 - V'(u^0) \tag{3.4}$$

where $S(u) = \int_0^L dx\, [\frac{1}{2}(\partial_x u)^2 + V(u)]$ is the equilibrium action.

Clearly, one has to know the equilibrium solutions of (3.4) that are the critical points of $S(u)$.

A complete study of these critical points can be found in ref. 8 or in ref. 11, where it is proven that their number and nature depend upon the parameter $\mu^{1/2}L$. It is clear that $u = 0$ is always a critical point. It can be proven that for small values of $\mu^{1/2}L$ it is the only one present.

The main results are summarized in the following.

Proposition 3.3. Let $N\pi < L\sqrt{\mu} < (N+1)\pi$, where N is an integer. Then:

(i) $S(u)$ has exactly $2N + 1$ critical points $\pm\phi_1 \cdots \pm\phi_N$, $\phi_{N+1} = 0$. The function ϕ_n, $1 \leqslant n \leqslant N$, has n half-periods.

(ii) $S(\pm\phi_1) < \cdots < S(\pm\phi_N) < S(0) = 0$.

(iii) $\pm\phi_1$ are the absolute minima of S and thus are absolute stable equilibria. The other critical points are saddles. The unstable manifold associated with ϕ_n has exactly $(n-1)$ dimensions.

For a proof see ref. 8 or ref. 11.

The picture that comes out from the above result and from a Ventzel and Freidlin type of analysis of the behavior of u_t^ε when the noise is small ($\varepsilon \ll 1$) is the following one: if the initial datum u_0 belongs to the basin of attraction of one of the two absolute minima, say $+\phi_1$ (we are assuming $L\sqrt{\mu} > \pi$), then the system, during a finite time $T_0(u_0)$, independent of ε, typically goes to the vicinities (in the uniform norm) of ϕ_1 and then starts randomly oscillating nearby ϕ_1 until a very unlikely large fluctuation leads it to the vicinities of the other minimum $-\phi_1$. The probability of occurrence of this "tunneling event" in a given time interval T is of order $\exp(-2\Delta S/\varepsilon^2)$, $\Delta S = S(\phi_2) - S(\phi_1)$, and so the typical time we have to wait to see tunneling is of the order $\exp(2\Delta S/\varepsilon^2)$.

The basic results behind this analysis are contained, as we said, in ref. 8; the complete Ventzel and Freidlin picture has been recently obtained by Freidlin.[12] We now turn to the statement of the main result of this section, which represents the counterpart for Eq. (3.1) of Theorem 2.1. We first fix some notation: let $u_{t;t_0}^\varepsilon(x; u_0)$ be the solution of (3.3) with initial condition u_0 at time t_0. We put

$$\|u_{t;t_0}^\varepsilon(u_0)\|_\infty = \sup_{0<x<L} |u_{t;t_0}^\varepsilon(x; u_0)|$$

For any $\delta > 0$ we define the δ-neighborhoods of $\pm\phi_1$ as $C_\delta^\pm = \{u; \|u \pm \phi_1\|_\infty < \delta\}$ and we set

$$C_\delta = C_\delta^+ \cup C_\delta^-$$

Next we define the quantity

$$L(u_{t;t_0}^\varepsilon; u) = \lim_{n\to 0} \sup_{\|u-v\|_\infty < \eta} \frac{\|u_{t;t_0}^\varepsilon(u) - u_{t;t_0}^\varepsilon(v)\|_\infty}{\|u-v\|_\infty} \tag{3.5}$$

which plays the role of derivative of the solution u_t^ε with respect to the initial condition. Let, finally, for $L\sqrt{\mu} > \pi$, m_0 be the smallest eigenvalue of the self-adjoint operator $-\Delta_D + 3\lambda\phi_1^2 - \mu$ on $L^2([0, L])$.

With these notations our theorem reads as follows.

Theorem 3.1. Let $\mu^{1/2}L > \pi$; then for any $m < m_0$ there exist positive constants δ, ε_0, k, and t_0 such that $\forall \varepsilon < \varepsilon_0$,

(a) $$P(u_{t;0}^{\varepsilon}(u_0)\in C_\delta\ \forall u_0\in C_\delta;\ \sup_{u\in C_\delta} L(u_{t;0}^{\varepsilon};u)<e^{-mt})$$
$$\geqslant 1-e^{-k/\varepsilon^2}\qquad \forall t>t_0$$

(b) $$P(\sup_{u\in C_\delta} L(u_{t;0}^{\varepsilon};u)<e^{-mt}\ \forall t>t_0)\geqslant 1-e^{-k/\varepsilon^2}$$

Corollary 3.1. Let $\mu^{1/2}L$, ε_0, δ, k, m_0, and t_0 be as in Theorem 3.1. Then there exists $c>0$ such that

$$P\left(\sup_{u,v\in C_\delta^+}\frac{\|u_{t;0}^{\varepsilon}(u)-u_{t;0}^{\varepsilon}(v)\|_\infty}{\|u-v\|_\infty}<ce^{-mt}\ \forall t>t_0\right)\geqslant 1-e^{-k/\varepsilon^2}$$

and the same for C_δ^-.

As is clear in the corollary, the above results prove the joining of two field configurations close to the same equilibrium solution under the random evolution induced by (3.3). An interesting question is whether the joining takes place when the two initial conditions are close to different equilibrium solutions, say $u\in C_\delta^+$, $v\in C_\delta^-$ (see also ref. 7 for a thorough discussion of the same problem).

The answer is that u and v will eventually join exponentially fast in time but only after a time $T_0(\varepsilon)$ of order $\exp(V/\varepsilon^2)$ for a suitable constant $V>0$. The time $T_0(\varepsilon)$ is the typical time scale needed for either u or v to jump under the random evolution to the other minimum of the action.

Proposition 3.4. In the same assumptions and notations of Theorem 3.1 there exists $V>0$ such that if $T_0(\varepsilon)=\exp(V/\varepsilon^2)$, then

$$P\left(\sup_{u,v\in C_\delta}\frac{\|u_{t;0}^{\varepsilon}(u)-u_{t;0}^{\varepsilon}(v)\|_\infty}{\|u-v\|_\infty}<ce^{-mt}\ \forall t>T_0(\varepsilon)\right)\geqslant 1-e^{-k/\varepsilon^2}$$

Remark 1. The above results have been stated for $L\sqrt{\mu}>\pi$ only. In this case, by Proposition 3.3, there are two minima of the equilibrium action. The results extend, however, without problems to the case $L\sqrt{\mu}<\pi$, when only one minimum is present. More complicated is the case $L\sqrt{\mu}=\pi$. For this value of the parameter the null solution of (3.4) is the only critical point of the action $S(u)$ but the linearization of (3.4):

$$\partial_t u=\partial_{xx}u+\mu u$$

has a zero mode, i.e., $\inf(\lambda;\ \lambda\in\operatorname{spec}(-\Delta_D-\mu))=0$, where $\Delta_D-\mu$ is looked upon as an operator on $L^2[0,L]$.

This fact prevent us from using the simple arguments exposed below (see the proof of Lemma 3.1) and a more detailed analysis is required.

The finite-dimensional analogue of this problem was treated in detail in ref. 7; there it was proved that in a situation like the above one, the constant m appearing in Theorem 3.1 is still positive but becomes infinitesimal as $\varepsilon \to 0$. Although we do not discuss the details here, we believe that the approach and the techniques developed in ref. 7 for this kind of problem apply also to the infinite-dimensional case, so that Theorem 3.1 holds also for $L\sqrt{\mu} = \pi$ but with $m = m(\varepsilon)$; $m(\varepsilon) \to 0$ as $\varepsilon \to 0$.

3.2. Proof of the Results

Proof of Corollary 3.1. Let δ be as in Theorem 3.1. It is very easy to see that

$$\sup_{u_0, v_0 \in C_\delta^+} \frac{\|u_t^\varepsilon(u_0) - u_t^\varepsilon(v_0)\|_\infty}{\|u_0 - v_0\|_\infty} \leqslant \sup_{u_0 \in C_\delta} L(u_{t;0}^\varepsilon; u_0)$$

Thus the Corollary follows from Theorem 3.1(b).

Proof of Proposition 3.4. Using Corollary 3.1, it is sufficient to prove that there exists $V > 0$ such that

$$P(\|u_t^\varepsilon(-\phi_1) - u_t^\varepsilon(\phi_1)\|_\infty < e^{-mt}\ \forall t > T_0) > 1 - e^{-k/\varepsilon^2} \tag{3.6}$$

for some $K > 0$ and ε small enough, where $T_0 = \exp(V/\varepsilon^2)$. To prove (3.6), we first observe that

$$u_t^\varepsilon(x; -\phi_1) < u_t^\varepsilon(x; +\phi_1) \qquad \forall t > 0, \quad \forall x \in [0, L] \tag{3.7}$$

In fact, let $\phi(t, x) = u_t^\varepsilon(x; +\phi_1) - u_t^\varepsilon(x; -\phi_1)$; then ϕ satisfies the evolution equation

$$\partial_t \phi = \partial_{xx}\phi - V'(u_t^\varepsilon(\phi_1)) + V'(u_t^\varepsilon(-\phi_1)) = \partial_{xx}\phi - V'(\tilde{u})\phi \tag{3.8}$$

where $\tilde{u}(t, x)$ lies between $u_t^\varepsilon(x; \phi_1)$ and $u_t^\varepsilon(x; -\phi_1)$. Using the Feynman–Kac formula, we can write

$$\phi(x, t) = E_x \left\{ \phi(w_t, 0) \exp\left[-\int_0^t ds\, V''(u(s, w_s)) \right] \right.$$
$$\left. \times\ \chi(w_s \in [0, L]\ \forall s \leqslant t) \right\} \tag{3.9}$$

where w_t is the Brownian motion starting at x. The positivity of $\phi(x, t)$ follows now from that of $\phi(x, 0)$. We omit the details of the proof of (3.6) because it follows step by step the proof of Proposition 2.2 of ref. 7 for the one-dimensional case.

Proof of Theorem 3.1. The proof follows step by step the proof of Theorems 1.1 and 2.1 of ref. 7. Thus, we will be very sketchy and refer the reader to ref. 7 for a simple description of the general strategy. We now give some details.

For each $a > 0$, $m > 0$ let $l = l(a) = [\exp(a/\varepsilon^2)]$ and let

$$S_0 = S_0(\{W_t\}_{t>0})$$

be given by

$$S_0 = \{j \in \mathbf{N};\, u^\varepsilon_{jl;(j-1)l}(u_0) \notin C_\delta \text{ or } L(u^\varepsilon_{jl;(j-1)l};\, u_0) > e^{-ml} \text{ for some } u_0 \in C_\delta\} \tag{3.10}$$

So if $\{1, 2,..., N\} \cap S_0 = \varnothing$, then

$$u^\varepsilon_{Nl;0}(u_0) \in C_\delta \qquad \forall u_0 \in C_\delta \tag{3.11}$$

and

$$\sup_{u_0 \in C_\delta} L(u^\varepsilon_{Nl;0};\, u_0) < e^{-mNl} \tag{3.12}$$

The last inequality follows immediately from (3.15) and the bound

$$L(u^\varepsilon_{t+s;0};\, u_0) < L(u^\varepsilon_{t;s};\, u^\varepsilon_s(u_0))\, L(u^\varepsilon_{s;0}(u_0)) \tag{3.13}$$

We need now a probabilistic control of the "bad set" S_0. We have the following result.

Lemma 3.1. For any $m < m_0$, where m_0 is the smallest eigenvalue of $-\Delta_D + V''(\phi_1)$, there exists ε_0, δ, and $k = k(\delta)$, $a_0 = a_0(\delta)$ such that for any $a < a_0$

$$P(j \in S_0) < \exp(-k/\varepsilon^2) \qquad \forall \varepsilon < \varepsilon_0$$

Proof. Let us fix $m < m_0$. We will show that it is possible to choose $\delta > 0$ so small that

$$\begin{aligned} P_{t,t_0} &= P(u^\varepsilon_{t+t_0;t}(u_0) \in C_\delta\ \forall u_0 \in C_\delta \text{ and } \sup_{u_0 \in C_\delta} L(u^\varepsilon_{t+t_0;t};\, u_0) < e^{-mt_0}) \\ &> 1 - e^{-2k/\varepsilon^2} \end{aligned} \tag{3.14}$$

for any t and some fixed t_0 (e.g., $t_0 = 1$) and a constant $k = k(\delta, t_0)$. If $a_0 \leqslant k$, then the lemma follows from (3.14). In fact, we have

$$\begin{aligned} P(j \notin S_0) &\geqslant 1 - \sum_{i=0}^{l/t_0} (1 - P_{it_0, t_0}) \\ &\geqslant 1 - \exp\left(\frac{a}{\varepsilon^2}\right) \exp\left(-\frac{2k}{\varepsilon^2}\right) \geqslant 1 - \exp\left(-\frac{k}{\varepsilon^2}\right) \end{aligned}$$

if ε is small enough.

We turn to the proof of (3.14). By stationarity we consider $t=0$. As shown in ref. 8, the difference between the random field $u(x, t)$ and the classical solution u_t^0 of (3.4) with the same initial condition is bounded by

$$\|u_t^\varepsilon - u_t^0\|_\infty < e^{k_1 t} \sup_{0<s<t} \|\varepsilon W_t\|_\infty \tag{3.15}$$

for some $k_1 > 0$.

Furthermore,

$$u_t^0 \in C_{\delta/2} \qquad \forall u_0 \in C_\delta \tag{3.16}$$

provided t is large enough (see Appendix A). Thus, if

$$\sup_{t<t_0} \|\varepsilon W_{t_0}\|_\infty < \frac{\delta}{2} e^{-k_1 t_0} \tag{3.17}$$

$$u_{t_0;0}^\varepsilon(u_0) \in C_\delta \qquad \forall u_0 \in C_\delta \tag{3.18}$$

As shown in ref. 8, the probability of (3.17) is at least $1-\exp(-k_2/\varepsilon^2)$ for some k_2 and ε small enough. In order to estimate $L(u_{t_0;0}^\varepsilon; u_0)$, we compute for u_0, v_0 in C_δ the difference $\phi_t = u_t^\varepsilon(u_0) - u_t^\varepsilon(v_0)$. We get

$$\partial_t \phi = \partial_{xx}\phi - V''(h(x, t))\phi \tag{3.19}$$

where $h(x, t)$ is between $u_t^\varepsilon(x; u_0)$ and $u_t^\varepsilon(x; v_0)$. Thus, if both $u_t^\varepsilon(u_0)$ and $u_t^\varepsilon(v_0)$ belong to C_δ, we have

$$|V''(h(x, t)) - V''(\phi_1(x, t))| \leqslant \text{const} \cdot \delta \tag{3.20}$$

and, by (3.19) and the Feynman–Kac formula, we get

$$\|\phi_t\|_\infty < e^{\text{const} \cdot \delta t} \|e^{-Ht}\phi_0\|_\infty \tag{3.21}$$

where $H = -\Delta_D + V''(\phi_1)$. It is now easy to see that

$$\|e^{-Ht}\phi_0\|_\infty < \text{const} \cdot e^{-m_0 t}$$

and therefore if δ is small enough

$$\|\phi_{t_0}\|_\infty < e^{-mt_0} \|\phi_0\|_\infty \tag{3.22}$$

Using the arbitrariness of u_0, v_0, we obtain from (3.22)

$$\sup_{u_0 \in C_\delta} L(u_{t_0;0}^\varepsilon; u_0) < e^{-mt_0} \tag{3.23}$$

provided $u_{t_0;0}^\varepsilon(u_0) \in C_\delta$, $\forall u_0 \in C_\delta$.

In conclusion, the probability of the event appearing in (3.19) is bounded from below by

$$1 - e^{-k/\varepsilon^2}$$

Remark 2. It is important to observe that if $V(u)$ is a strictly convex function, i.e., $V''(u) > m$ for any u, then the bound (3.22) holds for any realization of the noise term εW_t. So in this case the proof of Theorem 3.1 becomes trivial.

From Lemma 3.1 we then conclude that for typical configurations of the noise εW_t the set S_0 consists of small clusters well isolated one from the other.

Following ref. 7, we introduce a sequence of sets S_k

$$S_0 \supseteq S_1 \cdots \supseteq \cdots S_k \supseteq \cdots$$

as follows. Let $d_k = \exp(\lambda^k)$, $\lambda > 1$; we set

$$S_{k+1} = S_k \backslash S_k^g \tag{3.24}$$

where $S_k^g = \bigcup_\alpha C_k^\alpha$ is the maximal union of clusters of sites in S_0 (not necessarily connected) such that (i) diam $C_k^\alpha < d_k$; (ii) $\text{dist}(C_k^\alpha; S_k \backslash C_k^\alpha) < 2d_{k+1}$; (iii) there exist $\bar{C}_k^\alpha = (\bar{\alpha}, \bar{\beta})$ such that $(n_0 - 1)\, d_k < \text{dist}(\partial \bar{C}_k^\alpha, C_k^\alpha) < n_0 d_k$; $\bar{C}_k^\alpha$ is $(k-1)$-admissible and

$$u^\varepsilon_{\bar{\beta}l:\bar{\alpha}l}(u_0) \in C_\delta \qquad \forall u_0 \in B_\delta$$

where $n_0 \in \mathbf{N}$; δ is as in Lemma 3.1 and

Definition. A set $\Lambda \subset \mathbf{N}$ is k-admissible if $\partial \Lambda \cap \bar{C}_j^\alpha = \varnothing$, $\forall j \leqslant k$.

The sets S_k are the "bad sets" on scale k in the following sense

Proposition 3.5. Let $\Lambda \subset \mathbf{N}$ be an interval of length L, $\Lambda = [n, n+L]$, such that (i) $\Lambda \cap S_{k+1} = \varnothing$, (ii) $L > d_{k+1}/5$, (iii) Λ is k-admissible. Then

(a) $\quad u^\varepsilon_{(n+L)l;nl}(u_0) \in C_\delta \qquad \forall u_0 \in C_\delta$

(b) $\quad \sup\limits_{u_0 \in C_\delta} L(u^\varepsilon_{(n+L)l;nl}; u_0) < e^{-m_{k+1} lL}$

where $m_{k+1} > m_k - \text{const} \cdot (d_k/d_{k+1}) > m(1-\beta)$ for any $0 < \beta < 1$ and λ large enough.

Proof. The proof is identical to the one of Proposition 1.1 in ref. 7.

As far as the probability of the event $\Lambda \cap S_k = \varnothing$ is concerned, we have the following result (see Proposition 1.2 of ref. 7).

Proposition 3.6. There exists $\varepsilon_0 > 0$ and for $\varepsilon < \varepsilon_0$ a function $\eta(\varepsilon)$ with $\eta(\varepsilon) \to \infty$ as $\varepsilon \to 0$ such that

$$P(\Lambda \cap S_k \neq \varnothing) < |\Lambda| \frac{P(0 \in S_0)^{1/2}}{d_k^{\eta(\varepsilon)}}$$

where $|\Lambda| = \# \{j \in \Lambda\}$.

Proof. As is shown in Proposition 1.2 of ref. 7, the result follows from the following lemma

Lemma 3.2 (see Lemma 1.5 in ref. 7). For any ρ and any $a' > 0$ there exists a constant $k(\rho)$ such that for any ε small enough

$$\sup_{u_0 \in C_\delta} P(\tau(u_0, 0, T) > \rho T) < \exp[-kT \exp(-\delta/\varepsilon^2)]$$

where $\tau(u_0, 0, T)$ is the time spent by the random field $u_t^\varepsilon(u_0)$ starting at u_0, outside the set C_δ up to time T.

The proof is given in Appendix B.

The rest of the proof of Theorem 3.1 is now the same as that of Theorem 2.1 of ref. 7.

Remark 3. In the proof of Theorem 2.1,[7] a considerable simplification comes from the uniform Lipschitz condition on the drift term b. This leads to the estimate

$$L(x_t^\varepsilon; x) \leqslant e^{kt} \qquad \forall t, x$$

where x_t^ε is the solution of

$$dx_t^\varepsilon = b(x_t^\varepsilon)\, dt + \varepsilon\, dw_t$$

In our case the drift is not uniformly Lipschitz (in any sense) also because of the quantic behavior of $V(u)$. However, we still have the estimate

$$L(u_t^\varepsilon; u) < e^{kt} \qquad \forall u$$

This is because if we write explicitly $V''(h(x, t))$ in (3.19) we get

$$\partial_t \phi = \partial_{xx} \phi - 3\lambda h^2 \phi + \mu \phi$$

which gives together with the Feynman–Kac formula

$$|\phi(x, t)| < e^{\mu t} \, \|e^{\Delta_D t} \phi_0\|_\infty < e^{\mu t} \, \|\phi_0\|_\infty$$

4. THE INFINITE-DIMENSIONAL CASE II. ANALYSIS OF THE TUNNELING TIME

We consider here the analog of the problem analyzed in Section 1 for the model introduced in the previous section. We adopt the same notation of the Sections 2 and 3.

For $\mu^{1/2}L > \pi$ let C_δ^- and C_δ^+ be the spheres of radius δ in the uniform topology centered at the minima $-\phi_1$ and $+\phi_1$, respectively. We fix $m < m_0$ and we choose δ so small that Theorem 3.1 applies and $\lim_{t \to +\infty} u_{t;u_0}^0 = \pm\phi_1$, $\forall u_0 \in C_\delta^\pm$.

We define for $u_0 \in C_\delta^+$ the time of tunneling τ_{u_0} as the stopping time given by

$$\tau_{u_0} = \inf\{t \geqslant 0; u_{t;u_0}^\varepsilon \in C_\delta^-\} \tag{4.1}$$

Then we have the following results.

Theorem 4.1. Let $\beta(\varepsilon)$ be such that

$$\sup_{u_0 \in C_\delta^+} P(\tau_{u_0} > \beta) = e^{-1}$$

Then

(a) $$\lim_{\varepsilon \to 0} \frac{E(\tau_{\phi_1})}{\beta(\varepsilon)} = 1$$

(b) $$\lim_{\varepsilon \to 0} P(\tau_{\phi_1} > \beta t) = e^{-t} \qquad \forall t \geqslant 0$$

Proof. The proof is patterned on the scheme given for the finite-dimensional case. As in Section 2, we will prove the following basic lemma.

Lemma 4.1. For any $t > 0$

$$\sup_{u_0 \in C_\delta^+} |P(\tau_{u_0} > \beta t) - P(\tau_{\phi_1} > \beta t)| = o(1)$$

Let now $f_\varepsilon(t) = P(\tau_{\phi_1} > \beta t)$. Then, exactly as in Theorem 2.2, one proves

$$f_\varepsilon(t+s) = f_\varepsilon(t) f_\varepsilon(s) + o(1) \tag{4.2}$$

From (4.2), part (b) follows immediately as in Section 2. To prove (a), let

$$g(t) = \sup_{u_0 \notin C_\delta^-} P(\tau_{u_0} > \beta t)$$

Then [see (2.20)] we have

$$g(2^k) \leqslant g(2)^{k-1} \tag{4.3}$$

and we have only to estimate $g(2)$. Let

$$\sigma_{u_0} = \inf\{t \geqslant 0;\ u^{\varepsilon}_{t;u_0} \in C^+_{\delta} \cup C^-_{\delta}\} \tag{4.4}$$

Then, by the strong Markov property,

$$g(2) \leqslant \sup_{u_0} P(\sigma_{u_0} > \beta) + \sup_{u_0} E\{\chi(\sigma_{u_0} < \beta)\, P(\tau_{u(\sigma_{u_0};u_0)} > \beta)\} \tag{4.5}$$

Using Lemma 4.1 and the definition of σ_{u_0} and of β, we get that the second term in the rhs of (4.5) is bounded by e^{-1} while the first one, using Lemma 1 of Appendix B, is estimated by

$$\sup_{u_0} \exp[-\beta \exp(-a/\varepsilon^2)]\, E \exp\{[\exp(-a/\varepsilon^2)]\, \sigma_{u_0}\}$$
$$\leqslant 2 \exp[-\beta \exp(-a/\varepsilon^2)] \tag{4.6}$$

for any $a > 0$ and ε sufficiently small. From the large-deviation estimates on the tunneling probability given in ref. 8 it follows immediately that $\beta(\varepsilon) > \exp(h/\varepsilon^2)$ for some $h > 0$. Thus the rhs of (4.6) is $o(1)$ if a is small enough.

In conclusion, by (4.3), (4.5), and (4.6) we get

$$g(t) \leqslant c^{-t}$$

for some constant $c > 0$ and therefore we can perform the limit $\varepsilon \to 0$ inside the integral

$$\frac{E(\tau_{\phi_1})}{\beta} = \int_0^\infty dt\, P(\tau_{\phi_1} > \beta t) \tag{4.7}$$

and get the result.

It remains to prove Lemma 4.1. Let

$$T_0 = \sup_{u_0 \in C^-_{2\delta}} \inf\{t \geqslant 0;\ u^0_{t;u_0} \in C^-_{\delta/2}\}$$

where $u^0_{t;u_0}$ is the deterministic solution of (3.4) starting at u_0. Such a time is clearly finite because of the results of ref. 8 (see Appendix A).

Lemma 4.2.

$$\sup_{u_0 \in C^+_{\delta}} P(\tau_{u_0} < 1/\varepsilon^2 \text{ or } (\beta t - T_0) \leqslant \tau_{u_0} \leqslant \beta t) = o(1)$$

Assuming the above lemma, we now prove Lemma 4.1.

We write for $u_0 \in C_\delta^+$

$$\begin{aligned} P(\tau_{u_0} > \beta t) - P(\tau_{\phi_1} > \beta t) &= E\chi(\tau_{u_0} > \beta t)\chi\left(\frac{1}{\varepsilon^2} < \tau_{\phi_1} < (\beta t - T_0)\right) \\ &\quad + E\chi(\tau_{\phi_1} > \beta t)\chi\left(\frac{1}{\varepsilon^2} < \tau_{u_0} < (\beta t - T_0)\right) + o(1) \end{aligned} \tag{4.8}$$

where we have used Lemma 4.2.

We estimate only the first term in the rhs of (4.8), the second one being completely equivalent.

If χ_g denotes the characteristic function of the event

$$\{\|u^\varepsilon_{t;u_0} - u^\varepsilon_{t;\phi_1}\|_\infty < e^{-mt}\ \forall t > T_0\}$$

then, using Theorem 3.1, we can find T_0 and ε so small that the first term in (4.8) can be written as

$$\begin{aligned} &E\chi(\tau_{u_0} > \beta t)\chi\left(\frac{1}{\varepsilon^2} < \tau_{\phi_1} < (\beta t - T_0)\right) + o(1) \\ &\quad \leqslant \sup_{u_0 \in C_{2\delta}^-} P(\tau_{u_0} > T_0) \leqslant \sup_{u_0 \in C_{2\delta}^-} P(\sup_{0 < t < T_0} \|u^\varepsilon_{t;u_0} - u^0_{t;u_0}\| > \delta/2) \end{aligned} \tag{4.9}$$

Using (3.20)–(3.22), the last term in the rhs of (4.9) is $o(1)$.

Proof of Lemma 4.2. Using the Markov property, we have

$$\begin{aligned} &P(\beta t - T_0 \leqslant \tau_{u_0} < \beta t) \\ &\quad \leqslant \sup_{u_0 \in C_\delta^+} P_{u_0}(\tau_{u_0} < T_0) + \sup_{u_0 \in C_\delta^+} P_{u_0}(u^\varepsilon_{\beta t - T_0;u_0} \notin C_\delta^+ \cup C_\delta^-) = o(1) \end{aligned}$$

because of the Theorem 3.1 and of the large-deviation estimate of ref. 8.

On the other hand, the same proof of Lemma 3.1 shows that

$$P(\tau_{u_0} < 1/\varepsilon^2) = o(1)$$

This concludes the proof.

Remark 4. Although the time of tunneling is one of the most physically interesting stopping times of the model, from a probabilistic point of views one can imagine other rare events which take place on a time scale exponentially large in $1/\varepsilon^2$. Let, for example, $V(u)$ be a single-well potential, e.g., $V(u)=u^2$, with $V(u)\geqslant 0$, $V(0)=0$, $V''(0)=0$. Then in this case $u\equiv 0$ is the only critical point and there is no tunneling phenomenon. If we let

$$\tau_0=\inf\{t>0;\ \|u^{\varepsilon}_{t;0}\|_\infty=1\}$$

then we can try to prove the analog of Theorem 4.1 for τ_0.

However, the proof of Lemma 4.1 seems in this case more problematic, for the following reason. In order to prove that

$$\sup_{\|u_0\|_\infty<\delta} |P(\tau_{u_0}>\beta t)-P(\tau_0>\beta t)|=o(1)$$

we would argue as follows. Since $\tau_{u_0}\gg 1/\varepsilon^2$ with large probability $\forall u_0$, $\|u_0\|_\infty<\delta$, and since

$$\|u^{\varepsilon}_{t;u_0}-u^{\varepsilon}_{t;0}\|_\infty\leqslant e^{-mt} \qquad \forall t>t_0$$

again with large probability, we have that when, e.g., $\tau_{u_0}>\tau_0$, then $u^{\varepsilon}_{\tau_0;u_0}$ has a uniform norm greater than $1-\exp(-m/\varepsilon^2)$. In the finite-dimensional case this would imply that

$$\tau_0<\tau_{u_0}<\tau_{u_0}+e^{-3m/2\varepsilon^2}$$

thus proving the lemma. The proof was based on the simple remark that on a time scale $\exp(-3m/2\varepsilon^2)$ the drift term $b\,dt$ is negligible with respect to the noise term $\varepsilon\,dw_t$. In the infinite-dimensional case, however, we cannot immediately neglect the drift $d^2/dx^2-V'(u)$, since the solution u^{ε}_t is not differentiable and a more careful analysis is required.

If we denote by $\bar{u}(x)\equiv u^{\varepsilon}_{\tau_0;u_0}(x)$, then we have from Eq. (3.3)

$$u^{\varepsilon}_{\Delta t+\tau_0;u_0}-\bar{u}=eW_{\Delta t}+O(\Delta t)+(g\bar{u})(\Delta t) \tag{4.10}$$

with $\Delta t\simeq\exp(-3m/2\varepsilon^2)$.

One can show that

$$\varepsilon W_{\Delta t}\approx(\Delta t)^{1/4-\eta} \qquad \forall\eta>0$$

while

$$x\to\bar{u}(x,\Delta t)$$

is Hölder continuous with exponent $<1/2$. This implies that the first and third terms in the rhs of (4.10) are both of order $(\Delta t)^{1/4-\eta}$ unless particular cancellations occur in the integral $(g\bar{u})$. Since the third term relaxes to zero, the finite-dimensional argument breaks down.

Another way to rephrase the above problem is to imagine the field $u_t^\varepsilon(x)$ as the profile of a chain of coupled harmonic oscillators subjected to random kicks $\varepsilon\alpha(t, x)$ and to a force $-V'(u)$. The above result is equivalent to saying that, due to the roughness of the profile as a function of x, the force exerted by the other oscillator in x is comparable on a short time scale to the random kicks $\varepsilon\alpha$.

This fact prevents us from proving, for example, that when the field u_t^ε reaches a certain level (in the uniform topology), then at a time immediately after, it overcomes that level.

It is very likely that a frequency analysis is required in order to solve this problem.

APPENDIX A

Let $\|\cdot\|_2$ and $\|\cdot\|_1$ denote the L_2 and H_1 norms, respectively, in $[0, L]$, namely,

$$\|u\|_2^2 = \int_0^L dx\, u^2(x)$$

$$\|u\|_1^2 = \int_0^L dx \left(\frac{du}{dx}\right)^2 + \|u\|_2^2$$

We have

$$\|u_{t;u_0}^0 - \phi_1\|_1^2 \leqslant \|u_{t;u_0}^0 - \phi_1\|_2^2 + \left(\frac{1}{\sqrt{t}}\|u_0 - \phi_1\|_2 + c\int_0^t \frac{ds}{(t-s)^{1/2}\|u_{t;u_0}^0 - \phi_1\|_2}\right)^2 \tag{A.1}$$

[see ref. 8, Eq. (8.8)]. Moreover [see ref. 8, (8.6)]

$$\|u_{t;u_0}^0 - u_0\|_2 \leqslant e^{ct}$$

for some $c > 0$ and therefore

$$\|u_{t;u_0}^0 - \phi_1\|_1^2 \leqslant \text{const} \cdot te^{2ct}\,\|\phi_1 - u_0\|_2^2$$

i.e.,

$$\|u_{1;u_0}^0 - \phi_1\|_1^2 \leqslant \text{const} \cdot \delta \qquad \text{if} \quad \|\phi_1 - u_0\|_\infty < \delta \tag{A.2}$$

Since

$$\|u_{t;u_0}^0 - \phi_1\|_\infty \leqslant L^{1/2}\,\|u_{t;u_0}^0 - \phi_1\|_1$$

in order to prove (3.21) it is sufficient to show that

$$\|u^0_{t;u_0} - \phi_1\|_1 \leqslant \delta/2 \quad \forall u_0; \qquad \|u_0 - \phi_1\|_1 < \delta$$

provided δ is small enough and t sufficiently large.

This follows from the results of ref. 8.

APPENDIX B. PROOF OF LEMMA 3.2

The proof of the lemma follows that of the analogous result in finite dimension but the argument is even easier using the fact that in our case the time needed for the classical solution $u^0_{t;u_0}$ to reach the ball

$$B_R = \{u \in C_D;\ \|u\| \leqslant R\}$$

is bounded uniformly in the sup-norm of the initial condition and in R provided R is large enough. This is a consequence of the quartic behavior of the potential $V(u)$ at infinity. A simple proof of this fact goes as follows:

$$\frac{d}{dt}\|u^0_t\|_2^2 = 2\int_0^L dx\,[u^0_t \Delta_D u^0_t - u^0_t V'(u^0_t)]$$

$$\leqslant -2\int_0^L [\lambda(u^0_t)^4 - \mu(u^0_t)^2]\,dx \leqslant -2\frac{\lambda}{L}\|u^0_t\|_2^4 + 2\mu\,\|u^0_t\|_2^2 \tag{B.1}$$

because

$$\int_0^L u^2(x)\,dx \leqslant \left[\int_0^L u^4(x)\,dx\right]^{1/2} L^{1/2}$$

A simple calculation shows that (B.1) implies

$$\sup_{u_0} \inf\left\{t;\ \|u^0_t\|_2^2 < \frac{2\mu L}{\lambda}\right\} \leqslant C(\lambda, \mu, L) \tag{B.2}$$

since

$$\|u^0_t\|_1 \leqslant \|u^0_{t-1}\|_2 + \text{const}\cdot\int_{t-1}^t ds\,(t-s)^{-1/2}\,\|u^0_s\|_2$$

we get (B.2) for the H_1-norm and thus, by (A.3), for the sup-norm.

Let now for $\delta' > \delta$

$$\begin{aligned}
\tau_1 &= \inf\{t > 0;\ u^\varepsilon_{t;u_0} \notin C_\delta\}\\
\sigma_1 &= \inf\{t > \tau_1;\ u^\varepsilon_{t;u_0} \in C'_\delta\}\\
&\ \vdots\\
\tau_i &= \inf\{t > \sigma_{i-1};\ u^\varepsilon_{t;u_0} \notin C_\delta\}
\end{aligned} \tag{B.3}$$

and

$$\nu_t = \inf\{n \geqslant 0; \sigma_\nu > t\}$$

Clearly we have

$$\begin{aligned}
&\sup_{u_0} P(\tau(u_0, 0, T) > \rho T) \\
&\quad \leqslant P\left(\sum_{i=1}^{\nu} (\sigma_i - \tau_i) > \rho T\right) \\
&\quad \leqslant e^{-\beta_\rho T} \sum_n P(\nu_T = n)^{1/2} \left(\sup_{u_0} E_{u_0} e^{2\beta(\sigma_1 - \tau_1)}\right)^{n/2} \qquad \text{(B.4)}
\end{aligned}$$

for any $\beta > 0$. In the derivation of (B.4) we used the strong Markov property together with the exponential Chebyshev inequality. As far as the random variables ν_t and $(\sigma_1 - \tau_1)$ are concerned, we can prove the following result.

Lemma 1. There exists $\beta_0 > 0$ such that for any $a > 0$ if $\beta < \beta_0 \exp(-a/\varepsilon^2)$, then

$$\sup_{u_0} E_{u_0} e^{2\beta(\sigma_1 - \tau_1)} \leqslant 2$$

for any ε small enough.

Lemma 2. For any $a > 0$ any $n > [\exp(-a/\varepsilon^2)]\, T$ and ε sufficiently small

$$\sup_{u_0} P_{u_0}(\nu_t = n) \leqslant e^{-Kn/\varepsilon^2}$$

for some $K > 0$.

The two lemmas together with (B.4) show that for any β as in Lemma 1

$$\sup_{u_0} P(\tau(u_0, 0, T) > \rho T) \leqslant \exp(-\beta\rho T/2)$$

if ε is small enough. This conclude the proof of the lemma.

The proofs of Lemma 1 and Lemma 2 are identical to those of the finite-dimensional case [see (B.32) and (B.26) of ref. 7] and are omitted.

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